On some properties of circulant matrices

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Matrix

Definition

A **matrix** is an $m \times n$ array of numbers a_{ij} where $i = 1, \dots, m$, $j = 1, \dots, n$ of the form

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & & & \\ \vdots & & & \ddots & \vdots \\ & & & & \\ a_{m1} & \cdots & & & a_{mn} \end{bmatrix},$$

The set of matrices of size $m \times n$ is denoted by $\mathbb{M}_{mn}(\mathbb{C})$. If $m = n$ A is called **square matrix**.

Definition

A square matrix A is called a **diagonal matrix** if each of its non-diagonal element is zero. That is $a_{ij} = 0$ if $i \neq j$ and $a_{ii} \neq 0$.

Example

(1) $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

(2) $$\begin{bmatrix} i & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

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Toeplitz Matrix

Definition

An $n \times n$ matrix $T_n = [t_{ij}] \in \mathbb{M}_{nn}(\mathbb{C})$ is **Toeplitz Matrix** if $t_{ij} = t_{j-i}$ for $i, j = 0, 1, \ldots, n-1$, i.e., a matrix of the form

$$T_n = \begin{bmatrix} t_0 & t_1 & t_2 & \cdots & & t_{n-1} \\ t_{-1} & t_0 & t_1 & & & \\ t_{-2} & t_{-1} & & & & \\ \vdots & & & \ddots & & \vdots \\ & & & & t_0 & t_1 \\ t_{-(n-1)} & & & \cdots & t_{-1} & t_0 \end{bmatrix}.$$

Examples of Toeplitz Matrices

(1) $\begin{bmatrix} 4 & 5 & 1 & 0 & 0 \\ 7 & 4 & 5 & 1 & 0 \\ i & 7 & 4 & 5 & 1 \\ 0 & i & 7 & 4 & 5 \\ 0 & 0 & i & 7 & 4 \end{bmatrix}$ banded Toeplitz Matrix,

(2) $\begin{bmatrix} 14 & 3i & 0 & \cdots & 0 \\ 1 & 14 & 3i & \ddots & \vdots \\ 0 & 1 & 14 & 3i & 0 \\ \vdots & \ddots & 1 & 14 & 3i \\ 0 & \cdots & 0 & 1 & 14 \end{bmatrix}$ tridiagonal matrix,

(3) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{bmatrix}$ circulant matrix.

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Circulant matrix

Definition

An $n \times n$ matrix $C = [c_{ij}] \in \mathbb{M}_{nn}(\mathbb{C})$ is a **circulant matrix** if $c_{i,j} = c_{k,l}$, where $i, j = 0, 1, \ldots, n-1$ and $j - i = l - k (\mod n)$, i.e., if it has the form below

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & & c_{n-1} \\ c_{n-1} & c_0 & c_1 & & & \\ c_{n-2} & c_{n-1} & & & & \\ \vdots & & & \ddots & & \vdots \\ & & & & c_0 & c_1 \\ c_1 & & & \cdots & c_{n-1} & c_0 \end{bmatrix}.$$

Eigenvalues and eigenvectors

Definition

A scalar $\psi \in \mathbb{C}$ is called **an eigenvalue** of the $n \times n$ matrix $A \in \mathbb{M}_{nn}(\mathbb{C})$ if there is a nontrivial solution $y \in \mathbb{C}^n$ of

$$Ay = \psi y.$$

Such an y is called **an eigenvector** corresponding to the eigenvalue ψ.

Theorem

A scalar ψ is an eigenvalue of an $n \times n$ matrix $A \iff \psi$ satisfies the characteristic equation

$$\det(A - \psi I) = 0.$$

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Example

Let $T \in \mathbb{M}_{nn}(\mathbb{C})$.

$$T = \begin{bmatrix} 1+i & 0 & 0 \\ 0 & -4 & -3 \\ 0 & 1 & 0 \end{bmatrix}.$$

The characteristic equation is

$$\det(T - \psi I) = \det\left(\begin{bmatrix} 1+i & 0 & 0 \\ 0 & -4 & -3 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} \psi & 0 & 0 \\ 0 & \psi & 0 \\ 0 & 0 & \psi \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 1+i-\psi & 0 & 0 \\ 0 & -4-\psi & -3 \\ 0 & 1 & -\psi \end{bmatrix} = 0$$

$$(1+i-\psi)(-4-\psi)(-\psi) + 3(1+i-\psi) = 0$$

$$\psi_1 = -3, \quad \psi_2 = 1+i, \quad \psi_3 = -1.$$

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For each eigenvalue ψ, we have $(T - \psi I)y = 0$, where y is the eigenvector associated with eigenvalue ψ.
For $\psi_1 = -3$

$$(T - \psi I)y = \begin{bmatrix} 1 + i + 3 & 0 & 0 \\ 0 & -4 + 3 & -3 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{cases} (4 + i)x_1 = 0 \\ -x_2 - 3x_3 = 0 \\ x_2 + 3x_3 = 0 \end{cases} \implies \begin{cases} x_1 = 0 \\ x_2 = -3x_3 \\ x_3 \in \mathbb{C} \end{cases} \implies v_1 = (0, -3, 1)$$

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$$(T - \psi I)y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -4-1-i & -3 \\ 0 & 1 & -1-i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{cases} 0 = 0 \\ (-5-i)x_2 + (-3)x_3 = 0 \\ x_2 + (-1-i)x_3 = 0 \end{cases} \implies \begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{cases} \implies v_2 = (0, 0, 0)$$

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Eigenvalues and eigenvectors of circulant matrices.

The eigenvalues $\psi \in \mathbb{C}$ and the eigenvectors $y \in \mathbb{C}^n$ of C are the solutions of $Cy = y\psi$.

$$\begin{bmatrix} c_0 & c_1 & \cdots & & c_{n-1} \\ c_{n-1} & c_0 & \cdots & & c_{n-2} \\ \vdots & & \ddots & & \vdots \\ c_{n-1} & & & \cdots & c_0 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} \psi \ y_0 \\ \psi \ y_1 \\ \vdots \\ \psi \ y_{n-1} \end{bmatrix}.$$

$$\begin{aligned} c_0 y_0 + c_1 y_1 + \ldots + c_{n-1} y_{n-1} &= \psi y_0 \\ c_{n-1} y_0 + c_0 y_1 + \ldots + c_{n-2} y_{n-1} &= \psi y_1 \\ &\vdots \\ c_1 y_0 + c_2 y_1 + \ldots + c_0 y_{n-1} &= \psi y_{n-1} \end{aligned}$$

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$$c_0 y_0 + c_1 y_1 + \ldots + c_{n-1} y_{n-1} = \psi y_0$$

$$c_{n-1} y_0 + c_0 y_1 + \ldots + c_{n-2} y_{n-1} = \psi y_1$$

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$$\sum_{k=0}^{m-1} c_{n-m+k} y_k + \sum_{k=m}^{n-1} c_{k-m} y_k = \psi \; y_m,$$

$$\sum_{k=0}^{m-1} c_{n-m+k} y_k + \sum_{k=m}^{n-1} c_{k-m} y_k = \psi \, y_m, \text{ where } \; m = 0, 1, \ldots, n-1.$$

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$$\sum_{k=0}^{m-1} c_{n-m+k} y_k = \sum_{k=n-m}^{n-1} c_k y_{k+m-n} \;\text{ and}$$

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$$\sum_{k=0}^{n-1-m} c_k y_{k+m} + \sum_{k=n-m}^{n-1} c_k y_{k+m-n} = \psi\ y_m$$

Guess $y_k = \rho^k$ where $\rho \neq 0$, then

$$\sum_{k=0}^{n-1-m} c_k \rho^{k+m} + \sum_{k=n-m}^{n-1} c_k \rho^{k+m-n} = \psi \; \rho^m.$$

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$$\sum_{k=0}^{n-1-m} c_k \rho^k + \rho^{-n} \sum_{k=n-m}^{n-1} c_k \rho^k = \psi$$

If $\rho^{-n} = 1$, we have

$$\psi = \sum_{k=0}^{n-1} c_k \rho^k, \quad \text{and} \quad y = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 \\ \rho \\ \vdots \\ \rho^{n-1} \end{bmatrix}, \; y \in \mathbb{C}^n.$$

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Let $\varepsilon := e^{2\pi imk/n} \in \mathbb{C}$ be n- th roots of unity, for $n \in \mathbb{N}$. Then $U \in \mathbb{M}_{nn}(\mathbb{C})$ is matrix of the form

$$U = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \varepsilon & \varepsilon^2 & \dots & \varepsilon^{n-1} \\ 1 & \varepsilon^2 & \varepsilon^4 & \dots & \varepsilon^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \varepsilon^{n-1} & \varepsilon^{2(n-1)} & \dots & \varepsilon^{(n-1)(n-1)} \end{bmatrix}.$$

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$U \in \mathbb{M}_{nn}(\mathbb{C})$ - **unitary matrix** $\iff UU^* = U^*U = I$.

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Theorem

Let $C = [c_{i,j}] \in \mathbb{M}_{nn}(\mathbb{C})$ - circulant matrix, and $U \in \mathbb{M}_{nn}(\mathbb{C})$ - unitary matrix . Let $\Psi = diag(\psi_0, \psi_1, \ldots, \psi_{n-1})$ diagonal matrix with eigenvalues on diagonal. Then

$$C = U \Psi U^*.$$

Proof

Let $C = [c_{i,j}] \in \mathbb{M}_{nn}(\mathbb{C})$ - circulant matrix, $U \in \mathbb{M}_{nn}(\mathbb{C})$ - unitary matrix.
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$$CU = \frac{1}{\sqrt{n}} \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & & c_{n-2} \\ & c_{n-1} & & & & \\ \vdots & & \ddots & \ddots & & \vdots \\ & & & & c_0 & c_{-1} \\ c_1 & \cdots & & & c_{n-1} & c_0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \varepsilon & \varepsilon^2 & \dots & \varepsilon^{n-1} \\ & & & & \\ \vdots & \vdots & & \ddots & \vdots \\ & & & & \varepsilon^{(n-2)(n-1)} \\ 1 & \varepsilon^{n-1} & \varepsilon^{2(n-1)} & \dots & \varepsilon^{(n-1)(n-1)} \end{bmatrix}$$

$$= \frac{1}{\sqrt{n}} \begin{bmatrix} a_{0,0} & a_{0,1} & a_{0,2} & \cdots & & a_{0,n-1} \\ a_{1,0} & a_{1,1} & & \cdots & & a_{1,n-1} \\ & & & & & \\ \vdots & & \ddots & \ddots & & \vdots \\ & & & & & \\ a_{n-1,0} & \cdots & & & a_{n-1,n-2} & a_{n-1,n-1} \end{bmatrix} = \frac{1}{\sqrt{n}} \left[a_{l,m}\right],$$

where $l, m = 0, \cdots, n-1$.

$$[a_{l,0}] = \sum_{k=0}^{n-1} c_k$$

$$[a_{l,1}] = \sum_{k=0}^{n-l-1} c_k \varepsilon^{k+l} + \sum_{k=n-l}^{n-1} c_k \varepsilon^{k-n+l} = \sum_{k=0}^{n-1} c_k \varepsilon^{k+l}$$

$$\vdots$$

$$[a_{l,n-1}] = \sum_{k=0}^{n-l-1} c_k \varepsilon^{(n-1)(k+l)} + \sum_{k=n-l}^{n-1} c_k \varepsilon^{(n-1)(k-n+l)} = \sum_{k=0}^{n-1} c_k \varepsilon^{(n-1)(k+l)}$$

Because $\varepsilon^n = 1$ the above equations are equal.

$$U\Psi = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \varepsilon & \varepsilon^2 & \dots & \varepsilon^{n-1} \\ \vdots & \vdots & & \ddots & \vdots \\ & & & & \varepsilon^{(n-2)(n-1)} \\ 1 & \varepsilon^{n-1} & \varepsilon^{2(n-1)} & \dots & \varepsilon^{(n-1)(n-1)} \end{bmatrix} \begin{bmatrix} \psi_0 & 0 & \cdots & & 0 \\ 0 & \psi_1 & & & \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & & 0 & \psi_{n-1} \end{bmatrix}$$

$$= \frac{1}{\sqrt{n}} \begin{bmatrix} \psi_0 & \psi_1 & \cdots & \psi_{n-1} \\ \psi_0 & \psi_1\varepsilon & & \psi_1\varepsilon^{n-1} \\ & \psi_1\varepsilon^2 & & \\ \vdots & & \ddots & \vdots \\ \psi_0 & \psi_1\varepsilon^{n-1}\dots & & \psi_{n-1}\varepsilon^{(n-1)^2} \end{bmatrix}.$$

Because $\psi_m = \sum_{k=0}^{n-1} c_k e^{-2\pi imk/n}$ we have $CU = U\Psi$, and because U-unitary matrix we have $C = U\Psi U^*$.

Theorem

Let $C = [c_{i,j}] \in \mathbb{M}_{nn}(\mathbb{C})$, $B = [b_{i,j}] \in \mathbb{M}_{nn}(\mathbb{C})$ are circulant matrices with eigenvalues

$$\psi_m = \sum_{k=0}^{n-1} c_k e^{-2\pi imk/n}, \quad \beta_m = \sum_{k=0}^{n-1} b_k e^{-2\pi imk/n}, respectively.$$

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where $\Omega = diag(\psi_m + \beta_m)$.

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